Preparing for University Calculus

The information in this document is taken from "*Preparing for University Calculus*"[1], a booklet created by the APICS Committee on Mathematics and Statistics*, and edited by our own Robert Dawson, Professor in the Department of Math and Computing Science here at SMU.

*The Atlantic Provinces Council on the Sciences (APICS) is currently known as Science Atlantic. They can be found here: http://scienceatlantic.ca/

Worked examples and practice problems (with answers) are provided for each of the following topics, in the following order.

Work at your own pace, without a calculator. *Remember – calculators are not permitted when writing the MATH placement test.*

Arithmetic
Algebra
Inequalities and Absolute Values
Functions
Polynomials
Algebra with Fractions
Rationalizing Numerators or Denominators
Linear Graphs
Graphs
Exponents and Roots
Logarithms
Geometry and Basic Trigonometry
Trigonometric Identities
Problem Solving
More Difficult Problems *(Ans. not provided)*

Other important high school topics not covered in this document include: Probability, linear algebra, and coordinate geometry of conic sections.

Some Useful Facts

Algebraic identities

$$a^2 - b^2 = (a+b)(a-b) \qquad (a+b)^2 = a^2 + 2ab + b^2$$
$$a^3 - b^3 = (a^2 + ab + b^2)(a-b) \qquad a^3 + b^3 = (a^2 - ab + b^2)(a+b)$$

Exponential identities

$$a^x b^x = (ab)^x \qquad a^x a^y = a^{x+y}$$
$$(a^b)^c = a^{bc} \qquad a^{-b} = 1/a^b = (1/a)^b$$
$$\sqrt[a]{b^a} = \left(\sqrt[a]{b}\right)^a = b \qquad \sqrt[a]{b} = b^{1/a}$$
$$a^{\log_a(b)} = b \qquad \ln(a) = \log_e(a), \quad e = 2.71828\ldots$$
$$\log_a(b) + \log_a(c) = \log_a(bc) \qquad \log_a(b) - \log_a(c) = \log_a(b/c)$$
$$\log_a(b)\log_b(c) = \log_a(c) \qquad \log_a(1/b) = -\log_a(b)$$

Trigonometric facts

Let a triangle have sides of length a, b, c opposite angles A, B, C. Then

Pythagoras' Theorem: If C is right, $a^2 + b^2 = c^2$.

Sine Law: $\sin(A)/a = \sin(B)/b = \sin(C)/c$

Cosine Law: $c^2 = a^2 + b^2 - 2ab\cos(C)$

Trigonometric identities

$$\tan(\alpha) = \sin(\alpha)/\cos(\alpha) \qquad \cot(\alpha) = \cos(\alpha)/\sin(\alpha)$$
$$\cot(\alpha) = 1/\tan(\alpha)$$
$$\sec(\alpha) = 1/\cos(\alpha) \qquad \csc(\alpha) = 1/\sin(\alpha)$$
$$\sin^2(\alpha) + \cos^2(\alpha) = 1$$
$$\sec^2(\alpha) - \tan^2(\alpha) = 1 \qquad \csc^2(\alpha) - \cot^2(\alpha) = 1$$
$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$
$$\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$
$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$
$$\tan(A+B) = \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)} \qquad \tan(A-B) = \frac{\tan(A)-\tan(B)}{1+\tan(A)\tan(B)}$$

Arithmetic

Let's start with the basics!

This material, which is covered in elementary school and junior high, is very important. It is fundamental to everything that follows.

Expectations:

• You should be able to do basic arithmetic without a calculator, including operations on fractions, negative numbers, and decimals.

• You should be able to compute simple powers and roots.

Examples:

1. Find $\frac{1}{3} + \frac{2}{5}$.

   Solution: To add (or subtract) fractions, you must first find a common denominator:

$$\frac{1}{3} + \frac{2}{5} = \frac{1 \cdot 5}{3 \cdot 5} + \frac{2 \cdot 3}{5 \cdot 3} = \frac{5}{15} + \frac{6}{15} = \frac{11}{15}$$

2. Convert 1.25 to a fraction in lowest terms. **Solution:** To convert a decimal to a fraction, write as a fraction over a power of 10, then reduce.

$$1.25 = \frac{125}{100} = \frac{5 \cdot 25}{4 \cdot 25} = \frac{5}{4}$$

   (HINT: Memorizing some common equivalences between decimals and fractions, such as $1/2 = 0.5$, $1/3 = 0.333\ldots$, etc., could save you some time here!)

Arithmetic

Practice Problems:

Remember to practice under the same conditions as the test ….. No calculator.

1. Find $\frac{2}{3} + \frac{3}{5}$
2. Find $\frac{3}{4} - \frac{1}{5}$
3. Find $\frac{1}{3} \times \frac{6}{5}$ in lowest terms.
4. Convert 3.125 to a fraction in lowest terms.
5. Convert 37/25 to a decimal.
6. Convert 17/10 to a decimal.
7. Find $0.0001 \times 0.01/0.001$
8. Find 1.23×0.1
9. Find $1.005 + 9.995$
10. Find $1 - (-2)$
11. Find $\sqrt{64}$
12. Find $20^3/20^5$ as a decimal.

answers

(1) $\frac{19}{15}$ (2) $\frac{11}{20}$ (3) $\frac{2}{5}$ (4) $\frac{25}{8}$ (5) 1.48 (6) 1.7 (7) 0.001
(8)0.123 (9) 11.0 (10) 3 (11) 8 (12) 0.0025

Algebra

Basic Algebra
is closely related to Arithmetic

Expectations:
You should….

• Know the basic rules for addition, subtraction, multiplication, division, and exponents

• Be aware of the operations, such as division by 0 and taking the square root of a negative number, that cannot be done within the real number system

• Be able to solve a simple equation

• Simplify an algebraic expression

• Evaluate an expression by plugging-in values

Examples:

1. Simplify $\frac{a^7b^3}{a^4b^4}$.
   Solution: By basic rules of exponents
$$\frac{a^7b^3}{a^4b^4} = a^{7-4}b^{3-4} = a^3b^{-1} = \frac{a^3}{b}$$

2. Solve $ax + 4 = 2x - a$ for x when $a = 5$.
   Solution: Substituting $a = 5$ and adding, subtracting and dividing like quantities on both sides of the equation we reduce to an equation which is its own solution:
$$\begin{aligned} 5x+4 &= 2x-5 \\ 5x+4-2x &= 2x-5-2x \\ 3x+4-4 &= -5-4 \\ 3x\left(\frac{1}{3}\right) &= -9\left(\frac{1}{3}\right) \\ x &= -3 \end{aligned}$$
   so $x = -3$.

Algebra

Practice Problems:

Good skills in algebra are very important for success in calculus

1. Simplify $(a^4/a^2)^2$
2. Simplify $abc - acb + acd - ace$, factoring if possible .
3. Simplify $\frac{\frac{y}{x} + x}{\frac{2}{x}}$.
4. Simplify $(a^2 + a^3 + a^4)/a$, factoring if possible.
5. Solve $3x + 2 = x + 2$ for x.
6. If $x = 5$ and $y = 7$, find $x^2 - xy + 3y$.
7. Simplify $\frac{a^2 - (-a^2)}{a^2}$
8. If $a = 5$ and $b = 2$, find $a - b^2 + 2ab$
9. Solve $x^5 + 2x + 3 = x^5 - x$.
10. If $x = 3$ and $xy = 1$, find y.
11. Simplify $(a^2/a^{-2})(b^{-2}/b^2)$.
12. Solve $a + x + 2 = a - x + 4$.

answers

(1) a^4 (2) $ac(d-e)$ (3) $\frac{y-x^2}{2}$ (4) $a(1+a+a^2)$ (5) $x = 0$ (6) 11
(7) 2 (8) 21 (9) $x = -1$ (10) $y = 1/3$ (11) a^4/b^4 or $(a/b)^4$ (12) $x = 1$

Inequalities & Absolute Values

Expectations:
You should....
- be able to solve simple inequalities and perform algebraic operations with them
- know which operations reverse inequalities and which ones preserve them
- understand interval notation, including open, closed, half-open intervals, and intervals with limits at infinity
- know how to compute an absolute values
- do simple algebra using the absolute function

Examples:

1. Solve $\frac{7-2x}{3} \leq 4$.

   Solution: $\frac{7-2x}{3} \leq 4$ implies $7-2x \leq 12$ and $-2x \leq 5$.

   Thus $x \geq -\frac{5}{2}$, or $x \in [-\frac{5}{2}, \infty)$.

2. Solve $x^2 + 3x - 10 \leq 0$.

   Solution:

   Factoring the left side, $(x+5)(x-2) \leq 0$. The corresponding equation $(x+5)(x-2) = 0$ has solutions -5 and 2 which divide the real line into three intervals: $(-\infty, -5)$, $(-5, 2)$, $(2, \infty)$.

   On each of these intervals we determine the sign of the factors as follows:

Interval	$x+5$	$x-2$	$(x+5)(x-2)$
$(-\infty, -5)$	$-$	$-$	$+$
$(-5, 2)$	$+$	$-$	$-$
$(2, \infty)$	$+$	$+$	$+$

   From the table we find that $x^2 + 3x - 10 \leq 0$ on the interval $x \in [-5, 2]$.

3. Solve $|4x+5| > 9$.

   Solution: $|4x+5| > 9$ is equivalent to
   $4x + 5 > 9$ or $4x + 5 < -9$
   $4x > 4$ or $4x < -14$
   $x > 1$ or $x < -\frac{7}{2}$;
   which can be written as $x \in (-\infty, -\frac{7}{2}) \cup (1, \infty)$.

Inequalities & Absolute Values

Practice Problems:

1. If $y > x$, $x \geq w$, and $w < z$, which of the following must hold?
   (a) $y > w$, (b) $y < w$, (c) $y > z$, (d) $y < z$, (e) none of these.
2. For what values of x does $(x-2)^2 \geq 0$?
3. If $a > b$, can we conclude that:
   (a) $a^2 > b^2$ always; (b) $a^2 > b$ always; (c) $a^2 > b^2$ if $b > 0$;
   (d) $a^2 \geq b^2$ always; (e) none of these are true.
4. For what values of x is $1/(1+x) > -1$?
5. For what values of y is $y^2 > 0$?
6. For what values of y is $y^2 \geq 2$?
7. For what values of x is $|x-3| \leq 1$?
8. Which of the following intervals contains the point 0?
   (a)$(-\infty, 0)$ (b)$(-1, 1)$ (c)$(0, \infty)$
   (d)all of a,b,c (e) a and c only
9. Which of the following intervals contains the point 0?
   (a)$(-\infty, 0]$ (b)$[-1, 1]$ (c)$[0, \infty)$
   (d)all of a,b,c (e) a and c only
10. Find $|3 - |3-6||$
11. Simplify $x^2 - 2|x^2|$.
12. Solve $-1 < 2x - 5 < 7$.
13. Solve $x^2 - 3x + 2 > 0$.
14. Solve $|2x-3| \leq 5$.
15. Solve the equation $\frac{|x+3|}{|2x+1|} = 1$.

answers

(1) a (2) all x (or $-\infty, \infty$)) (3) c (4) $(-\infty, -2) \cup (-1, \infty)$
(5) $y \neq 0$ (6) $(-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty)$ (7) $[2, 4]$ (8) b (9) d (10) 0 (11) $-x^2$
(12) $x \in (2, 6)$ (13) $x \in (-\infty, 1) \cup (2, \infty)$ (14) $x \in [-1, 4]$ (15) $x = -\frac{4}{3}, 2$

Functions

Expectations:

You should…

• understand the concept of "function" and "inverse function"

• know how to compute the composition of two or more functions

• be able to determine the range and domain of a simple function

Examples:

1. Find the domain of the functions defined by:
(a) $f(x) = \frac{1}{x^2-1}$ (b) $g(x) = \sqrt{4-x}$

Solution

(a) The domain of $f(x) = \frac{1}{x^2-1}$ is the set of all values of x such that $x^2-1 \neq 0$; that is $x \neq \pm 1$ or $x \in \{(-\infty,-1) \cup (-1,1) \cup (1,\infty)\}$

(b) The domain of $g(x) = \sqrt{4-x}$ is the set of all values of x such that $4-x \geq 0$; that is $x \leq 4$ or $x \in (-\infty, 4]$.

2. If $f(x) = 3x+5$, find $\dfrac{f(x+h)-f(x)}{h}$

Solution

$$\frac{f(x+h)-f(x)}{h} = \frac{[3(x+h)+5]-[3x+5]}{h} = \frac{3x+3h+5-3x-5}{h}$$
$$= \frac{3h}{h} = 3$$

3. If $f(x) = \sqrt{x-1}$ and $g(x) = x^2$, find:
(a) $f(g(x))$ (b) $g(f(x))$

Solution

(a) $f(g(x)) = f(x^2) = \sqrt{x^2-1}$; this has domain $(-\infty,-1] \cup [1,\infty)$.

(b) $g(f(x) = g(\sqrt{x-1}) = (\sqrt{x-1})^2$
This is $x-1$ on the domain $[1,\infty)$ but is undefined elsewhere.

Functions

Practice Problems:

1. If $f(x) = 1/x$, find the largest possible domain (within the real numbers) on which f could be defined.

2. If $f(x) = 1/(x+1)$, find the domain of f.

3. If $g(x) = x^2 + 2$, find $g(1+x)$

4. If $f(x) = x^2$, find the range of f.

5. Find the domain and range of f(x) $= \dfrac{|x|}{x}$

6. If $f(x) = \dfrac{1}{x+1}$, find (a) $f(x^2)$; (b) $f(-2)$; (c)$f(\sqrt{x})$.

7. If $f(x) = 1/x$ and $g(x) = x^2$, find $f(g(a))$.

8. If $f(x) = 1/x$ and $g(x) = x^2$, find $g(f(2))$.

9. If $f(x) = 1/(x+1)$, find $f(f(x))$

10. If $f(x) = x^{-3}$, find the inverse function.

11. Which of the following functions is its own inverse:

    (a) $f(x) = x^2$, (b) $g(x) = -x$, (c)$h(x) = x$
    (d) each of f,g and h (e) g and h only

12. Which of the following functions has an inverse function?

    (a) $f(x) = x^3$, (b) $g(x) = x^5$, (c)$h(x) = x$
    (d) each of f,g and h (e) g and h only

13. Finds the inverse of $f(x) = \dfrac{x+1}{2x+1}$.

answers

(1) $x \neq 0$ or $(\infty, 0) \cup (0, \infty)$ (2) $x \neq -1$ or $(\infty, -1) \cup (-1, \infty)$
(3) $x^2 + 2x + 3$ (4) $[0, \infty)$ (5) Domain: $\neq 0$. Range: $\{-1, 1\}$ or ± 1 (6) (a) $\frac{1}{x^2+1}$
(b) -1 (c) $\frac{1}{\sqrt{x}+1}$ (7) a^{-2} (8) $1/4$ (9) $\frac{x+1}{x+2}$ (10)$x^{-1/3}$ (11) e (12) d (13)$\frac{1-x}{2x-1}$

Polynomials

Expectations:

You should...

- know how to add, subtract, multiply, divide, and factor polynomials
- know special forms such as the difference of powers
- understand the connection between roots and factorizations
- be able to solve a quadratic equation using the quadratic formula
- Be able to work with a polynomial function of something nontrivial, e.g., factor $\sin(x)^2 + 2\sin(x) + 1$

Examples:

1. Expand: $(x^2-1)(x^2+3)$

Solution:

$$\begin{aligned}(x^2-1)(x^2+3) &= x^2(x^2+3) - 1(x^2+3) \\ &= x^2 \times x^2 + x^2 \times 3 - 1 \times x^2 - 1 \times 3 \\ &= x^4 + 3x^2 - x^2 - 3 \\ &= x^4 + 2x^2 - 3\end{aligned}$$

2. Expand: $(x-1)(x+1)(x^2+1)$

Solution:

$$\begin{aligned}(x-1)(x+1)(x^2+1) &= (x-1)(x+1)\ (x^2+1) \\ &= (x(x+1) - 1(x+1))\ (x^2+1) \\ &= (x^2+x-x-1)(x^2+1) \\ &= (x^2-1)(x^2+1) \\ &= x^2(x^2+1) - 1(x^2+1) \\ &= x^4 + x^2 - x^2 - 1 \\ &= x^4 - 1\end{aligned}$$

3. Solve: $y^2 - 7y + 12 = 0$

Solution: This problem can be solved in one of two ways: either by factoring the quadratic, or by using the quadratic formula. We observe that

$$\begin{aligned} y^2 - 7y + 12 &= 0 \quad \text{through factoring becomes} \\ (y-4)(y-3) &= 0 \end{aligned}$$

which implies that the zeroes are $y = 3$ and $y = 4$.

Polynomials

Practice Problems:

1. Solve: $x^2 - 5x + 3 = 0$.
2. Factorize $2x^2 + 5x + 2$.
3. Put $x^2 + 2x + 2$ in completed square form, and graph it.
4. Simplify: $(x^2 + 3)(x^2 + a) - x^4$.
5. If we divide $x^4 + x^3 + x^2 + x + 1$ by $x + 1$, what is the remainder?
6. Expand: $(x + 2)^4$.
7. Expand and simplify: $\dfrac{(x+1)^3 - (x-1)^3}{x}$
8. Expand and simplify: $\dfrac{(x+1)^3 + (x-1)^3}{x}$
9. Expand: $(x^2 - 1)(x^2 - x + 1)$
10. How many real solutions does $x^4 - 5x^2 + 6 = 0$ have?
11. Factorize: $x^4 - 13x^2 + 36 = 0$

answers

(1) $x = (5 \pm \sqrt{13})/2$ (2) $(2x+1)(x+2)$ (3) $(x+1)^2 + 1$ (4) $(3+a)x^2 + 3a$ (5) 1 (6) $x^4 + 8x^3 + 24x^2 + 32x + 16$ (7) $(6x^2+2)/(x)$ (8) $2x^2 + 6$ (9) $x^4 - x^3 + x - 1$ (10) 4 (11) $(x-3)(x+3)(x-2)(x+2)$

Algebra with Fractions

Expectations:

You should…

- be able to simplify a fractional expression
- convert a "stacked" fractional expression into a single one
- put fractional expressions over a common denominator
- be able to perform a partial fraction expansion

Examples

1. Simplify $\dfrac{(x-1)^2+4x}{x+1}$.

Solution: $\dfrac{(x-1)^2+4x}{x+1} = \dfrac{x^2-2x+1+4x}{x+1} = \dfrac{x^2+2x+1}{x+1} = x+1.$

2. Simplify $\dfrac{\frac{x-1}{x+2} - \frac{x-2}{x+1}}{x}$.

Solution: $\dfrac{\frac{x-1}{x+2} - \frac{x-2}{x+1}}{x} = \dfrac{\frac{(x-1)(x+1)-(x-2)(x+2)}{(x+1)(x+2)}}{x} = \dfrac{(x^2-1)-(x^2-4)}{x(x+1)(x+2)} =$
$\dfrac{3}{x(x+1)(x+2)}.$

These skills will be useful in finding various derivatives, simplifying derivatives and integrals, and in particular for the "partial fractions" techniques of integration

Algebra with Fractions

Expectations:

You should…

- be able to simplify a fractional expression
- convert a "stacked" fractional expression into a single one
- put fractional expressions over a common denominator
- be able to perform a partial fraction expansion

…Examples

3. Expand $\dfrac{x^2+1}{x^2-1}$ in partial fractions.

Solution: $$\frac{x^2+1}{x^2-1} = 1 + \frac{2}{x^2-1} = 1 + \frac{2}{(x+1)(x-1)}.$$

We want to write this as $1+\frac{A}{x+1}+\frac{B}{x-1}$ for suitable A, B. Putting the fractions over a common denominator, we have $A(x-1)+B(x+1) = 0x+2$; so $A=-1$, $B=1$, and

$$\frac{x^2+1}{x^2-1} = 1 - \frac{1}{x+1} + \frac{1}{x-1}.$$

These skills will be useful in finding various derivatives, simplifying derivatives and integrals, and in particular for the "partial fractions" techniques of integration

Algebra with Fractions

Practice Problems:

1. Simplify $\dfrac{1}{x+1} - \dfrac{1}{x-1}$.
2. Write $x + 1 + 1/x + 1/x^2$ in the form $P(x)/Q(x)$ where P,Q are polynomials.
3. Simplify $\dfrac{x+1}{x+3} - \dfrac{x}{x+2}$. Is this equal to 0 for any x?
4. Simplify $\dfrac{1/x}{1/x^2}$.
5. Simplify $\dfrac{\frac{1}{x} - \frac{1}{x+h}}{h}$.
6. Solve $\dfrac{x+1}{x-2} - \dfrac{x}{x-1} = 0$.
7. Simplify $\dfrac{1/x \ + \ 1/y}{1/x \ - \ 1/y}$.
8. Simplify $\dfrac{x}{x+1} - \dfrac{1}{x(x+1)}$.
9. Expand $\dfrac{1}{x^2+x-6}$ in partial fractions.
10. Expand $\dfrac{x^3+4x^2+4x+1}{x^2+2x}$ in partial fractions.
11. Expand $\dfrac{1}{x^3-x}$ in partial fractions.

answers

(1) $\frac{-2}{x^2-1}$ (2) $\frac{x^3+x^2+x+1}{x}$ (3) $\frac{2}{(x+3)(x+2)}$;no. (4) x (5) $\frac{1}{x(x+h)}$
(6) $x = 1/2$ (7) $\frac{y+x}{y-x}$ or $1 + \frac{2x}{y-x}$ (8) $\frac{x-1}{x}$ (9) $\frac{-1/5}{x+3} + \frac{1/5}{x-2}$ (10) $x + 2 - \frac{1/2}{x} + \frac{1/2}{x+2}$
(11) $\frac{1/2}{x+1} - \frac{1}{x} + \frac{1/2}{x-1}$

Rationalizing Numerators or Denominators

Expectations:
You should…

- know how to eliminate square (and other) roots from the numerator or denominator by an appropriate expression

Examples

1. Rationalize the denominator of $\dfrac{1}{\sqrt{a}}$

   Solution: By multiplying both numerator and denominator by $\sqrt{a}$ we obtain

$$\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{a}}\frac{\sqrt{a}}{\sqrt{a}} = \frac{\sqrt{a}}{a}$$

2. Rationalize the denominator of $\dfrac{1+\sqrt{x}}{2+\sqrt{x}}$

   Solution: Recall, when we multiply $(a+b)(a-b)$ we obtain a difference of squares a^2-b^2. So, if the denominator of a rational expression contains a constant added to a square root term, we can eliminate the root term by multiplying by the difference of the constant and the root term. In this case if we were to multiply both the numerator and the denominator by $2-\sqrt{x}$ we will obtain the desired result.

$$\begin{aligned}\frac{1+\sqrt{x}}{2+\sqrt{x}} &= \frac{(1+\sqrt{x})}{(2+\sqrt{x})}\frac{(2-\sqrt{x})}{(2-\sqrt{x})}\\ &= \frac{2-\sqrt{x}+2\sqrt{x}-x}{4-2\sqrt{x}+2\sqrt{x}-x}\\ &= \frac{2+\sqrt{x}-x}{4-x}\end{aligned}$$

This technique will be important in finding the derivatives of certain expressions involving roots

Rationalizing Numerators or Denominators

Expectations:
You should…

• know how to eliminate square (and other) roots from the numerator or denominator by an appropriate expression

…Examples

3. Rationalize the numerator of $\dfrac{1+\sqrt{x}}{2+\sqrt{x}}$

Solution: This time we multiply numerator and denominator by the conjugate of the numerator:

$$\frac{1+\sqrt{x}}{2+\sqrt{x}} = \frac{(1+\sqrt{x})}{(2+\sqrt{x})}\frac{(1-\sqrt{x})}{(1-\sqrt{x})} = \frac{1-x}{2-\sqrt{x}-x}$$

This technique will be important in finding the derivatives of certain expressions involving roots

Rationalizing Numerators or Denominators

Practice Problems:

1. Rationalize the denominator of $\dfrac{x+1}{\sqrt{x}}$
2. Rationalize the denominator of $\dfrac{a}{\sqrt{a}\sqrt[3]{b}}$
3. Rationalize the denominator of $\dfrac{1}{\sqrt{x}+\sqrt{y}}$
4. Rationalize the denominator of $\dfrac{a-\sqrt{x}}{b-\sqrt{x}}$
5. Rationalize the denominator of $\dfrac{1}{1+x^{1/3}+x^{2/3}}$
6. Rationalize the numerator of $\dfrac{\sqrt{x}+\sqrt{y}}{x}$
7. Rationalize the numerator of $\dfrac{a-\sqrt{x}}{b-\sqrt{x}}$
8. Rationalize the numerator of $\dfrac{\sqrt{x+1}-\sqrt{x}}{x}$
9. Rationalize the numerator of $\dfrac{\sqrt[3]{y+h}-\sqrt[3]{y}}{h}$

answers

(1) $\frac{x\sqrt{x}+\sqrt{x}}{x}$ (2) $\frac{\sqrt{a}\sqrt[3]{b^2}}{b}$ (3) $\frac{\sqrt{x}-\sqrt{y}}{x-y}$ (4) $\frac{ab+(a-b)\sqrt{x}-x}{b^2-x}$ (5) $\frac{x^{1/3}-1}{x-1}$

(6) $\frac{x-y}{x(\sqrt{x}-\sqrt{y})}$ (7) $\frac{a^2-x}{ab+(b-a)\sqrt{x}-x}$ (8) $\frac{1}{x(\sqrt{x+1}+\sqrt{x})}$ (9) $\frac{1}{(y+h)^{2/3}+(y+h)^{1/3}h^{1/3}+h^{2/3}}$

Linear Graphs

Expectations:

You should…

- be able to graph linear functions and inequalities
- Be able to determine the slope and intercept of a line from its equation – and vice versa
- Be able to determine where two lines meet
- Be able to use the negative reciprocal rule for orthogonal lines
- Find the distance between two points

Examples

1. Find the point of intersection of the lines $x = 2$ and $x + y = 5$.

   Solution: This does not require a graph to be drawn. Remember that the line (or curve) corresponding to an equation is the set of points (x, y) for which the equation is true; and the intersection of two lines or curves is the set of pairs (x, y) for which both equations are true.

   So we need to find a pair (x, y) such that $x = 2$ and $x + y = 5$. The first equation gives us the value of x immediately; plugging that into the other equation we get $2 + y = 5$, $y = 3$, so the answer is $(x, y) = (2, 3)$.

2. Find the slope of the line through the points $(-5, 0)$ and $(3, 2)$.

   Solution: The "rise" is $2 - 0 = 2$ and the "run" is $3 - (-5) = 8$. Slope = rise/run = $2/8 = 1/4$.

Many of these ideas will be conceptually important in calculus

Calculus deals a lot with slopes, tangent lines, secant lines, etc.

Linear Graphs

Expectations:

You should…

- be able to graph linear functions and inequalities
- Be able to determine the slope and intercept of a line from its equation – and vice versa
- Be able to determine where two lines meet
- Be able to use the negative reciprocal rule for orthogonal lines
- Find the distance between two points

Examples

3. Find the equation of the line through $(5, 0)$ and orthogonal to $x + 2y = 3$.

   Solution: First we find the slope of the given line $x + 2y = 3$. To find the slope we isolate y in the equation, rewriting it first as $2y = 3 - x$ and then as $y = \frac{3}{2} - \frac{1}{2}x$. Thus the given line has slope $m = -\frac{1}{2}$.

   The line we are looking for is orthogonal to the given line, so its slope is the negative reciprocal of the first line's slope. $-1/(-\frac{1}{2}) = 2$, so we are looking for a line with slope 2, hence with eqation $y = 2x + b$.

   Every point on that line satisfies that equation, so $0 = 2 \times 5 + b$. Solving, we find that $b = -10$, and our equation is

$$y = 2x - 10\,.$$

Many of these ideas will be conceptually important in calculus

Calculus deals a lot with slopes, tangent lines, secant lines, etc.

Linear Graphs

Practice Problems:

1. Find the slope of the line $2y = x - 2$
2. Give the equation of a line with slope 3, through $(0, 0)$.
3. If a line has slope $3/2$ and passes through $(2, 2)$, give its equation.
4. Find the distance from the point $(0, 3)$ to the point $(3, 0)$
5. Find the slope of the line through the points $(0, 3)$ and $(3, 0)$.
6. Find the point of intersection of the lines $y = 5 - x$ and $y = 2x + 2$
7. Find the equation of the line with slope $-1/2$ and y-intercept 3
8. Find the slope of a line orthogonal to the line $y = 3x + 5$.
9. Find the equation of the line through $(3, 2)$ and $(1, 0)$

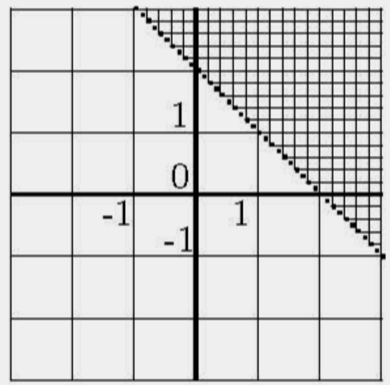
answers

(1) $1/2$ (2) $y = 3x$ (3) $y = \frac{3}{2}x - 1$ (4) $3\sqrt{2}$ or $\sqrt{18}$ (5) -1
(6) $(1, 4)$ (7) $y = 3 - x/2$, or $y = -\frac{1}{2}x + 3$ (8) $-1/3$ (9) $y = x - 1$ (10) e
(11) c (12) b

Linear Graphs

Practice Problems:

10. Which inequality is this the graph of?



(a) $y > 2$ (b) $x > y + 2$ (c) $x < y + 2$ (d) $x > 2$ (e) $x + y > 2$

11. Which of these lines is parallel to $2y = 6x - 1$?

(a) $2y = 5x - 1$ (b) $y = 6x + 1$ (c) $y = 3x + 1$
(a) $2y = 3x + 2$ (a) $2y = -6x + 1$

12. Which of these lines is orthogonal to $y = 3x - 1$?

(a) $y = -3x - 1$ (b) $y = -x/3 + 1$ (c) $y = x/3 + 1$
(d) $3x = y + 2$ (e) $y = 1 - 3x$

answers

(1) $1/2$ (2) $y = 3x$ (3) $y = \frac{3}{2}x - 1$ (4) $3\sqrt{2}$ or $\sqrt{18}$ (5) -1
(6) $(1, 4)$ (7) $y = 3 - x/2$, or $y = -\frac{1}{2}x + 3$ (8) $-1/3$ (9) $y = x - 1$ (10) e
(11) c (12) b

Graphs

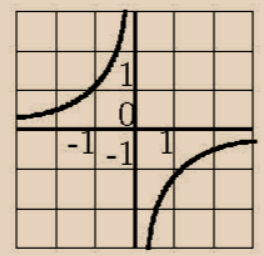
Expectations:

You should…

- be able to graph polynomials and rational functions, showing features such as:
  - zeros
  - y-intercept
  - horizontal asymptotes
  - vertical asymptotes
  - slant asymptotes
  - points of discontinuity
- Be able to read significant features from a graph

Examples

1. Which equation best fits the given curve?



(a) $y = x^2$
(b) $y = 1/x$
(c) $y = \sin(x)$
(d) $x = -1/y$
(e) $x = y^2$

Solution: As the graph has a vertical asymptote (here, $x = 0$), we can rule out (a),(c), and (e). Of the remaining two options, (b) has a graph consisting of points in the first and third quadrants (x and y are either both positive or both negative), while the equation (d) is satisfied by points such as $(1, -1)$ that are on the given curve. So we choose (d), which does indeed have all the right features.

A graph—for our purposes—should be drawn by determining the main features and joining them together with *smooth curves.*

Graphs

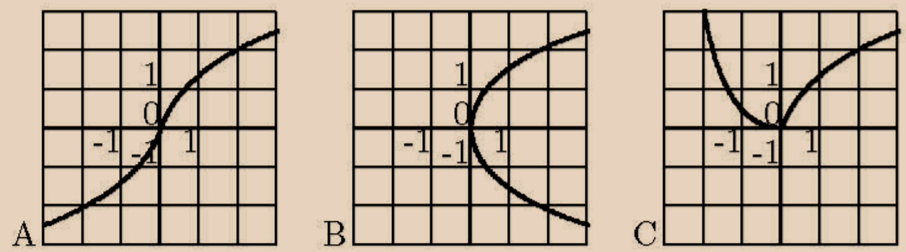
Expectations:

You should…

- be able to graph polynomials and rational functions, showing features such as:
  - zeros
  - y-intercept
  - horizontal asymptotes
  - vertical asymptotes
  - slant asymptotes
  - points of discontinuity
- Be able to read significant features from a graph

Examples

2. Which of the following is the graph of a function $y = f(x)$?



(a) A only
(b) B only
(c) C only
(d) all three
(e) A and B
(f) A and C

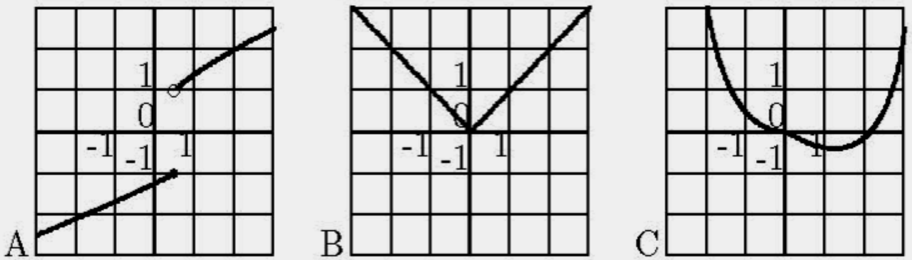
Solution: (f). Graph B does not have a unique y value for every x, and the others do. The smoothness of graph B, and the fact that it appears to be a parabola, are not relevant. (It *is* the graph of an equation $x = f(y)$.)

In your calculus course you will learn to extend these graphing skills by adding other features, such as maxima, minima, and points of inflection.

Graphs

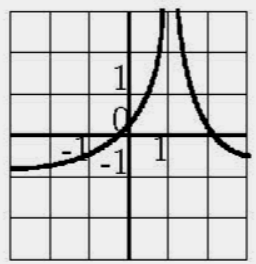
Practice Problems:

1. Which of the following is the graph of a function $y = f(x)$?



(a) A only
(b) B only
(c) C only
(d) all three
(e) B and C
(f) A and C

2. Which equation best fits the given curve?



(a) $y = 1/(x-1)^2 - 1$
(b) $y = 1 - 1/x$
(c) $y = 1/(x+1)^2 - 1$
(d) $y = 1/x^2$
(e) $y = 1/(x+1)^2 + 1$

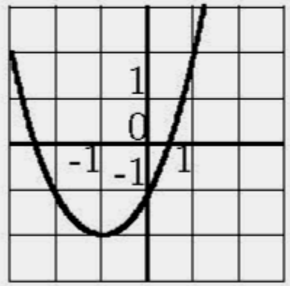
answers

(1) d (2) a (3) d (4) e

Graphs

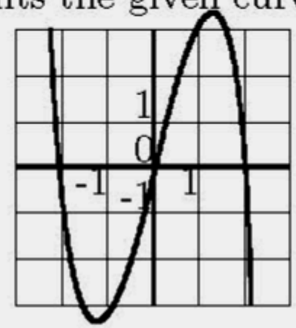
Practice Problems:

3. Which equation best fits the given curve?



(a) $y = x^2 - 2$
(b) $y = (x + 1)^2$
(c) $y = x^2 + 2x - 1$
(d) $y = x^2 - 2x - 1$
(e) $y = x^2 + 2x - 2$

4. Which equation best fits the given curve?



(a) $y = x^3$
(b) $y = -x^3$
(c) $y = x^3 + 4x$
(d) $y = x^3 - 4x$
(e) $y = 4x - x^3$

answers
(1) d (2) a (3) d (4) e

Exponents and Roots

Expectations:

You should…

- know the basic identities for exponents and roots, and be able to solve equations and derive their identities
- Be able to convert a reciprocal to a negative power, or a root to a fractional power

Examples

1. Find $4^{18}/4^{16}$.

   Solution: $4^{18}/4^{16} = 4^{18-16} = 4^2 = 16$.

2. Simplify $\sqrt{a^{7/2}a^{5/2}a^{3/2}a^{1/2}}$.

   Solution: First we simplify the expression under the square root sign:

   $$a^{7/2}a^{5/2}a^{3/2}a^{1/2} = a^{7/2+5/2+3/2+1/2} = a^8 .$$

   So

   $$\sqrt{a^{7/2}a^{5/2}a^{3/2}a^{1/2}} = \sqrt{a^8} = a^4 .$$

3. Solve: $x^{-2} = 100$.

   Solution: Taking the reciprocal of both sides, $x^2 = 1/100$. Taking the square root of both sides, $x = 1/10$ or 0.1.

4. If $a^4 = 9$, find a^{-2}

   Solution: If $a^4 = 9$, $a^2 = 3$ and $a^{-2} = 1/3$.

These identities will be extremely important for differentiation and integration in your calculus course

Exponents and Roots

Practice Problems:

1. Simplify $\dfrac{\sqrt{x} \times x^2}{1/x^2}$

2. If $a^{3/2}a^{-2}\sqrt[3]{a} = a^b$, what is b?

3. Solve: $4^x 2^x = 64$.

4. Solve: $4^0 + 2^x = 9$.

5. Find $\left(\dfrac{1}{\sqrt{16}}\right)^{1/2}$

6. If $a^4 = b^8$, for real numbers a, b, which of the following must hold?
   (a) $a = b^2$ (b) $a = -b^2$ (c) $a = b^{-2}$
   (d) $a = b^2$ or $a = -b^2$ (e) none of these must hold.

7. $\sqrt{a^{16}b^{16}} =$ (a) $(ab)^2$ (b) $(ab)^4$ (c) $(ab)^8$ (d) $(ab)^{16}$
   (e) none of these.

8. If $a^b = (\sqrt[3]{a})^{12}$, what is b?

9. If $a^3 = b$, and $b^{1/6} = 10$, find a.

10. If $\sqrt[b]{a} = a^2$, find b.

answers

(1) $x^{9/2}$ (2) $-1/6$ (3) $x = 2$ (4) $x = 3$ (5) $1/2$ (6) d
(7) c (8) 4 (9) 100 (10) $1/2$

Logarithms

Expectations:

You should…

- know the definition of logarithms to various bases
- know their relation to powers and roots
- know the change of bas formula: $\log_a(b)\,\log_b(c) = \log_a(c)$

Examples:

1. Find $\log_5(125)$.
   Solution: We recognize 125 as the cube of 5, (or, in a pinch determine that fact by trial and error.) $5^3 = 125$, so $\log_5(125) = 3$.
2. Simplify $\log_2(3)\log_3(4)$
   Solution: Using the change-of-base rule, $\log_2(3)\log_3(4) = \log_2(4)$. Because $4 = 2^2$ we have $\log_2(4) = 2$.
3. What is the decimal logarithm of $10\sqrt{10}$?
   Solution: $10\sqrt{10} = 10^{3/2}$, so $\log_{10}(10\sqrt{10}) = 3/2$.

Logarithms are extremely important in many of the sciences. You must be able to manipulate them algebraically before you can begin to solve problems with them in your calculus course.

Logarithms

Expectations:

You should…

- know the definition of logarithms to various bases
- know their relation to powers and roots
- know the change of bas formula: $\log_a(b)\log_b(c) = \log_a(c)$

Examples:

4. If $\log_a(b^2) = 3$, find $\log_b(a^2)$
   Solution: If $\log_a(b^2) = 3$, then $\log_a(b) = 3/2$. By the change-of-base rule, $\log_b(a) = 2/3$ and so $\log_b(a^2) = 4/3$.

5. (Without a calculator or notes!) The natural logarithm of 10 is between:
   (a) 0 and 1 (b) 1 and 2 (c) 2 and 4 (d) 4 and 6 (e) 6 and 10
   Solution: It would be useful here to know $\ln(10) = 2.3025\ldots$, but you do not need to. If you just know that e, the base of the natural logs, is between 2 and 3, you can rule out (a) and (b) as too small (because $e^2 < 3^2 < 10$) and (d) and (e) as being too big (because $e^4 > 2^4 > 10$).

   (The actual value of e is $2.71828\ldots$)

Logarithms are extremely important in many of the sciences. You must be able to manipulate them algebraically before you can begin to solve problems with them in your calculus course.

Logarithms

Practice Problems:

1. Find $\log_{10}(0.001)$.
2. Find $\log_2(64) + \log_3(9)$
3. If $3^x = 7$, which is true?
   (a) $7 = \log_x(3)$ (b) $x = \log_7(3)$ (c) $3 = \log_x(7)$ (d) $7 = \log_3(x)$
   (e) $x = \log_3(7)$ (f) $3 = \log_7(x)$
4. $\log_3(5)\log_5(7)\log_7(9) =$
   (a)1 (b)2 (c)3 (d)4 (e) none of these
5. If $\log_a(25) = 4$, what is a?
   (a)1/5 (b)$\sqrt{5}$ (c)5 (d)$\log_2(5)$ (e) none of these
6. Find $\log_{100}(1,000,000)$
7. Find $\log_{100}(10)$
8. If $\log_{10}(2) \approx 0.301$, which of these is closest to $\log_{10}(2000)$?
   (a) 0.6 (b) 3 (c) 6 (d) 30 (e) 60 (f)300
9. If $\log_{10}(2) \approx 0.301$, which of these is closest to $\log_{10}(8)$?
   (a) 0.3 (b) 0.6 (c) 0.9 (d) 1.2 (e) 2.4 (f)6
10. If $\log_{10}(2) \approx 0.301$, which of these is closest to $\log_2(10)$?
    (a) 0.5 (b) 1 (c) 3 (d) 5 (e) 20 (f)50
11. Find $2^{\log_2(17)}$
12. Find $4^{\log_2(3)}$
13. If $\log_3(10) = K$, then $\log_9(10) =$
    (a)$2K$ (b)$K/3$ (c)$K/2$ (d)K^2 (e)$3K$ (f)$\sqrt{K}$

Answers

(1) -3 (2) 8 (3) e (4) b (5) b (6) 3 (7) 1/2 (8) b
(9) c (10)c (11) 17 (12) 9 (13) c

Geometry & Basic Trigonometry

Expectations:

Calculus does not use very advanced geometry, but you should be thoroughly familiar with similar triangles, Pythagoras' theorem, and parallel lines; and, from analytic geometry, the midpoint and distance formulae, and the "negative reciprocal" rule. Trigonometry is important in various branches of science, but especially in mathematics, physics, and engineering.

You should be able to convert between degrees and radians ($180° = \pi$ radians). You should know the definitions of the trig functions, and be able to use them to find sides and angles of triangles. You should know and be able to use the sine and cosine laws for triangles.

Most angles do not have trig functions that are easy to give as exact expressions rather than decimal approximations (and you will not be expected to do so), but you should know the trig functions of a few common angles, such as $0°$, $30°$, $45°$, $60°$,and $90°$. You should also know how to find the trig functions of angles outside the range $[0°, 90°]$ in terms of trig functions of angles in that range.

You should also be familiar with the inverse trigonometric functions. Note that although (for instance) $\sin^2(x)$ means $(\sin(x))^2$, and $\sin(x)^{-1}$ means $1/\sin(x)$, which is $\csc(x)$, $\sin^{-1}(x)$ means $\arcsin(x)$.

Geometry & Basic Trigonometry

Examples:

1. If the hypotenuse of a right triangle is 10, and one leg has length 8, how long is the other leg?
   Solution: Let the length of the unknown leg be x. By Pythagoras' theorem, $x^2 + 8^2 = 10^2$, so $x^2 = 100 - 64 = 36$ and $x = 6$.

2. Find the midpoint of the segment from the point $(2, 3)$ to $(8, -3)$.
   Solution: By the midpoint formula, the coordinates of the midpoint are $\left(\frac{2+8}{2}, \frac{3+-3}{2}\right)$ which simplifies to $(5, 0)$.

3. If two sides of a triangle have length 1 and 2, and the angle between them is $45°$, what is the length x of the remaining side?
   Solution: By the cosine law, $x^2 = 1^2 + 2^2 - 2(1)(2)\cos(45°) = 5 - 4(\sqrt{2}/2) = 5 - 2\sqrt{2}$; and therefore $x = \sqrt{5 - 2\sqrt{2}}$.
   We cannot simplify this further so we leave it in this form.

Practice Problems

1. If a triangle has sides 15, 20, and 25 units long, what is the measure in degrees of its largest angle?
2. If a rectangle has edge lengths 10 and 15, how long is its diagonal?
   (a) 25 (b) $\sqrt{25}$ (c) $\sqrt{125}$ (d) $\sqrt{225}$ (e) $\sqrt{325}$
3. Which of these triples could *not* be the edge lengths of a right triangle?
   (a) $(3,4,5)$ (b) $(12,16,20)$ (c) $(2,2,3)$ (d) $(12,13,5)$ (e) $(\sqrt{2},\sqrt{2},2)$
4. In the plane, what are the coordinates of the midpoint of the line segment whose ends are $(10,6)$ and $(4,10)$?
   (a) $(2,3)$ (b) $(3,2)$ (c) $(5,5)$ (d) $(7,8)$ (e) none of these are correct.
5. What is the distance between the points whose coordinates are $(-2,3)$ and $(1,-1)$?
   (a) 5 (b) 7 (c) $9/2$ (d) $\sqrt{7}$ (e) $\sqrt{5}$
6. Find the cosine of $225°$
7. Find the arctangent of 1, in degrees

answers

(1) $90°$ (2) e (3) c (4) d (5) a (6) $-1/\sqrt(2)$ or $-0.707\ldots$ (7) $45°$ (8) $\sqrt{3}$ (9) $75\pi/180$ or $5\pi/12$ (10) c (11) c (12) $1/\sqrt(3)$ (13) b (14) d

Geometry & Basic Trigonometry

Practice Problems

8. Find the cotangent of 30°
9. How many radians is 75°?
10. How many degrees is $3/2$ radians? (a)$2\pi/3$ (b)180 (c)$270/\pi$ (d)$\pi/120$ (e)$3\pi/2$
11. How many angles θ between 0° and 360° have $\sin(\theta) = 1/2$?
    (a)0 (b)1 (c)2 (d)3 (e)4
12. Find $\tan(\pi/6)$
13. The value of $\arcsin(0.5)$ is: (a) 0° (b) 30°
    (c) 45° (d) 60° (e) 90°
14. If $\cos(\theta) = 3/5$, then $\sin(\theta)$ can be
    (a) $4/5$ only (b)$5/4$ only (c) $5/4$ or $-5/4$
    (d) $4/5$ or $-4/5$ (e) $-5/4$ only

answers

(1) 90° (2) e (3) c (4) d (5) a (6) $-1/\sqrt{(2)}$ or $-0.707\ldots$ (7) 45° (8) $\sqrt{3}$ (9) $75\pi/180$ or $5\pi/12$ (10) c (11) c (12) $1/\sqrt{(3)}$ (13) b (14) d

Trigonometric Identities

Expectations:

There are many identities that are satisfied by the trigonometric functions. They are important in calculus because they are used to reduce the number of rules you have to learn. Note that $\cos^2(\alpha)$ means $(\cos(\alpha))^2$.

The identities for one angle can all be derived from the definitions of the six functions, via the relations $\tan(\alpha) = \sin(\alpha)/\cos(\alpha)$, $\cot(\alpha) = \cos(\alpha)/\sin(\alpha)$, $\sec(\alpha) = 1/\cos(\alpha)$ $\csc(\alpha) = 1/\sin(\alpha)$ and from the identities $\sin^2(\alpha)+\cos^2(\alpha) = 1$, $\sec^2(\alpha) - \tan^2(\alpha) = 1$, and $\csc^2(\alpha) - \cot^2(\alpha) = 1$.

There are also various identities for trigonometric functions of sums and differences of angles, and for double and half angles. These are aso useful in calculus.

Trigonometric Identities

Examples:

1. $\sin^2(\theta)\sec(\theta)\csc(\theta) =$
   (a) $\sin(\theta)$ (b) $\cos(\theta)$ (c) $\tan(\theta)$ (d) $\sec(\theta)$ (e) $\csc(\theta)$ (f) $\cot(\theta)$
   Solution:
   $$\begin{aligned}\sin^2(\theta)\sec(\theta)\csc(\theta) &= (\sin(\theta)\sec(\theta))\,(\sin(\theta)\csc(\theta)) \\ &= (\sin(\theta)(1/\cos(\theta)))\,(\sin(\theta)(1/\sin(\theta))) \\ &= (\tan(\theta))\end{aligned}$$
   so the answer is (c).
2. If $\sin(35°) = \cos(\beta)$, and $\beta \in [0°, 90°]$, find β.
   Solution: $\sin(\alpha) = \cos(90° - \alpha)$ for all α; so here $\beta = 55°$.
3. If $\cos(\alpha) = 0.3$, find $\cos(2\alpha)$.
   Solution: We do not need to know α for this! One form of the "double angle formula" for cosines is
   $$\cos(2\alpha) = 2\cos^2(\alpha) - 1$$
   from which we see that $\cos(2\alpha) = 2(0.3^2) - 1 = -0.82$.

Trigonometric Identities

Practice Problems

1. $\sin(2x) =$
   (a) $2\sin(x)$ (b) $\sin(x) + \cos(x)$ (c) $\cos(x)\sin(x)$
   (d) $2\cos(x) - 2\sin(x)$ (e) $2\cos(x)\sin(x)$

2. $\cot(\alpha)\cos(\alpha)\sin(\alpha) =$
   (a) $\cos(\alpha)\sin(\alpha)$ (b) $\sin^2(\alpha)$ (c) $\cos^2(\alpha)$ (d) $\cos(\alpha)$ (e) $\sin(\alpha)$

3. $\tan(\alpha)\cot(\alpha) =$
   (a) $\sin(\alpha)$ (b)$\cos(\alpha)$ (c) $\sin^2(\alpha)$ (d)$\cos^2(\alpha)$ (e) 1

4. If $\cos(\theta) = \sqrt(2)/2$, then $\tan(\theta) =$
   (a) 2 (b) 1 (c) 0 (d) 1/2 (e) ± 1

5. If $\tan(\theta) = 3$, then $\cot(\theta) =$
   (a) 3 (b) 1/3 (c) 0 (d) -3 (e) $-1/3$

6. If $\sin(\gamma) = 1/3$, $\sin(-\gamma) =$
   (a) 2/3 (b) 1/3 (c) $-1/3$ (d) $-2/3$ (e) none of these

7. If $\cos(\gamma) = 1/3$, $\cos(-\gamma) =$
   (a) 2/3 (b) 1/3 (c) $-1/3$ (d) $-2/3$ (e) none of these

8. If $\cos(\alpha) = 3/5$, find $\cos(2\alpha)$

9. How many angles θ between $0°$ and $360°$ have $\sin(\theta) = \cos(\theta)$?
   (a)0 (b)1 (c)2 (d)3 (e)4

10. How many angles θ between $0°$ and $360°$ have $\sin(\theta) = \sec(\theta)$?
    (a)0 (b)1 (c)2 (d)3 (e)4

answers

(1) e (2) c (3) e (4) e (5) b (6) c (7) b (8) $-7/25$ (9) c (10) a

Problem Solving

Expectations:

Solving applied problems ("word problems") using calculus uses many of the same skills as solving applied problems using algebra. In each case, you have to pick out the important numerical quantities — known or unknown — from the problem, and determine the relations between them. These yield a set of equations that must be solved to yield the desired quantity. You may also need to know certain quantities and relations that are *not* given in the problem.

Do **not** try to learn "the formula for each type of problem." A common mistake is to think that there is one formula for "the problem with the lawnmowers" and another for "the problem with the antifreeze". There are many different problems involving time to complete tasks, and many different problems involving mixtures. Instead, learn basic relations and heuristics, such as that when two agents (people, taps, etc) cooperate on a task, the *amount of work done* can be added and the *time taken* cannot.

Problem Solving

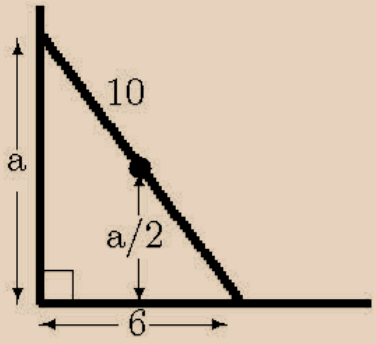
Examples:

1. A tap fills a 100-liter tank in 2 hours, if the tank starts empty and the drain is closed. If the drain is open, and the tap is off, the tank drains at a constant rate; starting full, it would empty in 3 hours. If the tank starts half-full at noon, when is it completely full?

   Answer: We combine the effects of the tap and drain by adding or subtracting work done per hour. The tap adds 100 ℓ in 2 hours, or $100/2\ \ell/\text{hr}$. The drain removes 100 ℓ in 3 hours, or $100/3\ \ell$. We need to ad 50ℓ to the tank; this takes $50\ell/(100/6\ \ell/\text{hr}) = 3\text{hr}$, so the tank is full at 3:00 PM.

2. If a $10m$ ladder leans against a vertical wall, with the foot of the ladder $6m$ from the wall, how high above the ground is the midpoint of the ladder?

   Answer: **Draw a diagram and label it.** Because the wall is vertical, and we assume the ground to be horizontal, the wall and ground make a right angle. Therefore the wall, ground, and ladder make a right triangle (with the ladder as hypotenuse) and Pythagoras' theorem applies. $a^2 + 6^2 = 10^2$, so $a = \sqrt{(100 - 36)} = 8$, and the height of the midpoint is $a/2$ or 4.



Problem Solving

Practice Problems:

1. If the area of a triangle is 100 cm^2 and its base is equal to twice its height, what is its base? (a) 5 cm (b) 10 cm (c) 20 cm (d) 50 cm (e) 200 cm
2. If the perimeter of a rectangle is 28 meters, and the diagonal is 10 meters, what is the area?
3. If the perimeter of a rectangle is 26 meters, and the area is 30 square meters, what is the diagonal?
4. A right triangle has area 30 square meters and perimeter 30 meters. What is its hypotenuse?
5. If a goat can eat the grass on a lawn in 1 day and a sheep can eat the grass on the same lawn in 2 days, how long would it take two goats and a sheep working together? Assume the grass is not growing.
(a) 2/5 of a day (b) 3/8 of a day (c) 1 day (d) 2 days (e) 4days

answers

(1) c (2) $24m^2$ (3) $\sqrt{109}$ m (4) 13 m (5) a (6) f (7) 3 kg
(8) d (9) 12:30 PM (10) 8:40 AM

Problem Solving

Practice Problems:

6. Joe started day trading with a certain amount of money. On the first day he doubled his money, and on the second day he lost 2/3 of what he had at the end of the first day. At the end of the second day he had \$ 100. How much did he start with?
   (a) \$50 (b) \$75 (c) \$80 (d) \$125 (e) \$133.33 (f)\$150

7. If a puppy weighs a third as much as a dog, and together they weigh 12 kg., how much does the puppy weigh?

8. A driver is driving 100 km. She drives the first 50 km at 50 km/hr and the second 50 km at 150 km/hr. How long does the trip take? (a) 30minutes (b) 50 minutes (c) 60 minutes (d) 80 minutes (e) 90 minutes

9. The towns of Ayton and Beaton are 60 km apart. At noon, Alice leaves Ayton and drives to Beaton at 80 km/hr, and Bob leaves Beaton and drives (on his tractor) to Ayton at 40 km/hr. At what time do they meet?

10. The towns of Ayton and Beaton are 60 km apart. At noon, Alice leaves Ayton and drives to Beaton at 90 km/hr. If Bob rides from Ayton to Beaton on his bicycle at 15 km/hr, when must he leave Ayton to get to Beaton at the same time that Alice does?

answers

(1) c (2) 24m^2 (3) $\sqrt{109}$ m (4) 13 m (5) a (6) f (7) 3 kg
(8) d (9) 12:30 PM (10) 8:40 AM

More Difficult Problems

A final note …

The material earlier in this booklet represents the minimum that you need to know before you begin university calculus. We hope that it won't be *all* that you know! You should also know something about linear algebra, geometry, statistics, and other areas of mathematics; you should have experience applying mathematics in other subjects; and you should be able to write clear explanations of what you know, and solve problems that require a certain amount of lateral thinking. In this section, we present some problems intended to challenge the stronger student. Such students may also want to look for problems in publications such as Crux Mathematicorum, or compete in mathematics contests.

More Difficult Problems

Some final practice (no answers given)

1. Find all solutions to $x^7 + 4x^5 + x^3 - 6x = 0$.
2. Explain why it is impossible for $x^{11} - 7x^7 + x^3 - 4x = 0$ to have an even number of real solutions. Find a general statement about polynomials that this is a general case of.
3. You want to know if $ax^8 + bx^6 + cx^4 + dx^2 + e$ has an odd or even number of real solutions, where a, b, c, d, e are real numbers (not all 0) that you don't know. You are allowed to "buy a letter" and find out its value for \$10 per letter . Explain how to find the answer for minimum cost.
4. Explain why $123abc567$ cannot be a perfect square (where a, b, c are unknown digits), without using a calculator or doing any lengthy calculations.
5. Given that $2^{10} = 1024$, show how to estimate the decimal logarithm of 2; and find the first digit and number of digits of 2^{100}.
6. Explain why we cannot define $0/0$ to be 1 without changing other rules of mathematics.
7. Find the area of a regular octagon, if its edge length is 1.
8. (a) If we put the biggest possible circle inside a square of radius 1, and the biggest possible square into that circle, what would its area be?

   (b) If we put the biggest possible sphere inside a cube of radius 1, and the biggest possible cube into that sphere, what would its volume be?

End of practice problems

Sample Mathematics Placement Test

Saint Mary's University

- The test contains a total of 40 multiple choice questions.
- Mark your answers (A, B, C, D or E) in the appropriate boxes below.
- You have 45 minutes to complete the test.
- **Calculators or other aids are not permitted.**

#	ANSWER
1.	
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#	ANSWER
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1. $\frac{4}{5} - \frac{3}{4} =$

(A) 1 **(B)** $\frac{1}{20}$ **(C)** $\frac{1}{16}$ **(D)** $\frac{1}{12}$ **(E)** 12

---

2. $0.125 \times 4 =$

(A) 55 **(B)** 0.55 **(C)** .055 **(D)** 0.45 **(E)** none of these

---

3. $((1-(2-1))-2) =$

(A) 0 **(B)** 1 **(C)** -1 **(D)** -2 **(E)** 2

---

4. $\dfrac{xy}{y - \frac{y}{x}} =$

(A) $\frac{y^2}{x-1}$ **(B)** $\frac{x^2}{x-1}$ **(C)** $\frac{y^2}{1-x}$ **(D)** $\frac{x}{1-x}$ **(E)** $\frac{x}{x-1}$

---

5. $\dfrac{1}{\sqrt{5}-\sqrt{3}} =$

(A) $\frac{\sqrt{5}-\sqrt{3}}{2}$ **(B)** $\frac{\sqrt{8}}{2}$ **(C)** $\frac{\sqrt{5}+\sqrt{3}}{2}$ **(D)** $\frac{\sqrt{5}-\sqrt{3}}{8}$

(E) $\frac{\sqrt{5}+\sqrt{3}}{8}$

---

6. If $P = \sqrt{\dfrac{\alpha+\beta^2}{M}}$, where $\alpha = 3$, $\beta = 5$, and $M = 7$, then $P =$

(A) $\frac{3}{2}$ **(B)** $\frac{\sqrt{3}}{2}$ **(C)** 2 **(D)** $\frac{1}{2}$ **(E)** $\sqrt{3}$

---

7. Consider only $(x-1)$, $(x+1)$ and $(x-2)$ as possible factors of x^3+x^2+x+1 . Of these only

(A) $(x-1)$ is a factor **(B)** $(x+1)$ is a factor **(C)** $(x-2)$ is a factor

(D) $(x-1)$ and $(x+1)$ are factors. **(E)** None of the preceding are true.

---

8. $(x^2+1)(x^5+x^3+1) =$

(A) $x^7+x^5+x^3+x^2+1$ **(B)** $x^7+x^6+x^5+x^4+x^3+x^2+x+1$

(C) $x^7+2x^5+2x^3+2x+1$ **(D)** $x^7+2x^5+x^3+x^2+1$

(E) $x^7+2x^5+2x^3+2x^2+1$

---

9. If $2x^2 - x = 1$, then $x =$

(A) $-\frac{1}{2}$ or 1 **(B)** 2 or 1 **(C)** $-\frac{1}{2}$ or 2

(D) 1 or -1 **(E)** $\frac{1}{2}$ or 1

---

10. If $x + 2y = 3$, and $2x - y = 3$, then (x, y)

(A) $= (1, 1)$ **(B)** $= (\frac{3}{2}, \frac{3}{2})$ **(C)** $= (\frac{9}{4}, -\frac{1}{4})$ **(D)** $= (\frac{9}{5}, \frac{3}{5})$

(E) = none of these

---

11. If $f(x) = x^4 - 2x^3 + x$, then $f(\frac{1}{2}) =$

(A) $\frac{x^4 - 2x^3 + x}{16}$ **(B)** $\frac{5}{16}$ **(C)** $\frac{1}{16}$ **(D)** 0 **(E)** $\frac{x^3 - x^2}{16}$

---

12. If $f(x) = 1 + x^2$, then $f(1 - x) =$

(A) $1-x-x^2$ **(B)** $2-x^2$ **(C)** $1-2x-x^2$ **(D)** x^2-2x+2

(E) none of these

---

13. If $f(x) = x^2 + x$, then $f(x - h) =$

(A) $x^2 - x + h$ **(B)** $x^2 + h^2 - x - h$ **(C)** $x^2 + h^2 - x + h$

(D) $x^2 + 2hx + h^2 - x + h$ **(E)** none of these

---

14. If $f(x) = x^2 + xb$, then $f(x + b) =$

(A) $x^2 + 2bx + b^2$ **(B)** $x^2 + 3bx + b^2$ **(C)** $x^2 + 3bx + 2b^2$

(D) $x^2 + xb + b$ **(E)** none of these

---

15. If $|x + 2| - 1 < 7$, then which of these follows?

(A) $x < 6$ **(B)** $x > -10$ **(C)** $x > -2$

(D) $-10 < x < 6$ **(E)** none of these

---

16. Let x be the length of the side of a square. If each side is decreased by 2 inches, the area of the square is decreased by 100 square inches. What is the area of the square after the sides are decreased?

(A) 526square inches **(B)** 426square inches **(C)** 476square inches
(D) The area cannot be determined from the information given
(E) None of the above is correct

---

17. $2^0 + 1^{-2} =$

(A) 1 **(B)** 2 **(C)** $2\frac{1}{2}$ **(D)** 3 **(E)** none of these

---

18. If $3^x = 5$, then

(A) $\log_3(5) = x$ **(B)** $\log_x(3) = 5$ **(C)** $\log_x(5) = 3$

(D) $\log_3(x) = 5$ **(E)** none of these are true

---

19. $\log_{10}(9) - \log_{10}(3) =$

(A) $\log_{10}(6)$ **(B)** $\log_9(3)$ **(C)** $\log_{10}(27)$ **(D)** $\log_{10}(\frac{1}{2})$

(E) none of these

---

20. $\frac{x-1}{x+1} - \frac{x-2}{x-1} =$

(A) $\frac{1-2x}{x^2-1}$ **(B)** $\frac{3x-2}{x^2-1}$ **(C)** 1 **(D)** $\frac{3+x}{x^2-1}$

(E) none of these

---

21. The function $p(x) = (x^2+1)(x-1)$

(A) changes sign three times
(B) changes sign twice
(C) changes sign once
(D) is always positive
(E) is always negative

---

22. The slope of the line passing through the points $(-1, 0)$ and $(1, 3)$ is

(A) $\frac{3}{2}$ **(B)** 3 **(C)** -2 **(D)** $\frac{2}{3}$ **(E)** 2

---

23. The slope of the line perpendicular to the line $2y = 3x + 1$ is

(A) -1 **(B)** $-\frac{2}{3}$ **(C)** $\frac{1}{3}$ **(D)** $-\frac{1}{3}$ **(E)** none of these

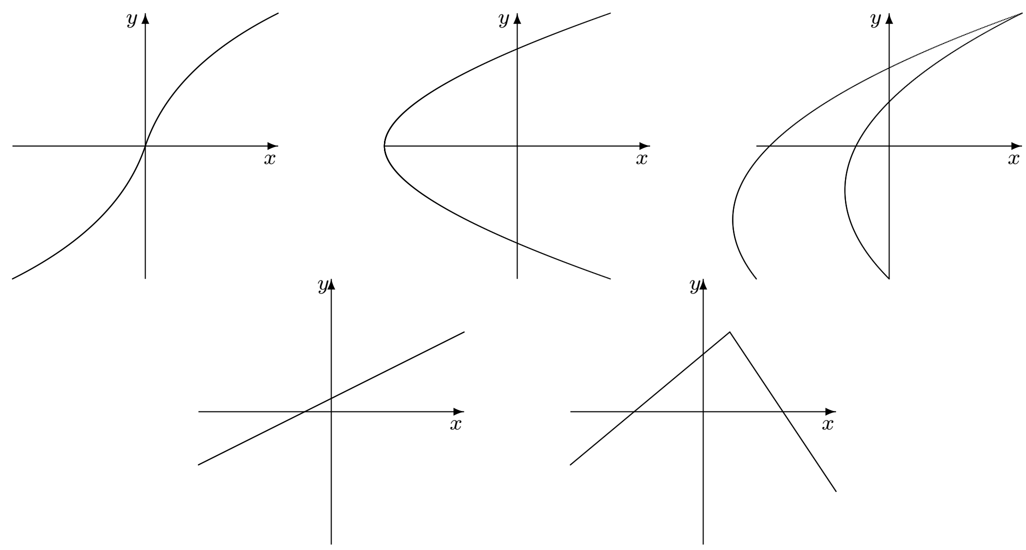
---

24. The distance between the points $(-1, 2)$ and $(5, -5)$ is

(A) 13 **(B)** $\sqrt{5}$ **(C)** 5 **(D)** $\sqrt{55}$ **(E)** $\sqrt{85}$

---

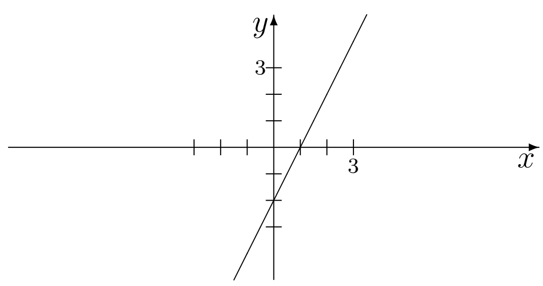
25. For the following set of graphs, which statement is true?



(A) They are all graphs of functions
(B) Exactly four of them are graphs of functions
(C) Exactly three of them are graphs of functions
(D) Exactly two of them are graphs of functions
(E) Exactly one of them is a graph of a function

---

26. Which equation has this line as its graph?



(A) $y = x - 1$ **(B)** $y = \frac{1}{2}x + 1$ **(C)** $x + y = 1$ **(D)** $y = 2x - 2$

(E) none of these

---

27. How many of the following equations represent straight lines?

$xy = 9$ $x^2 + y^2 = 4$ $x + 1 = y^2$ $x + y = 16$

(A) none **(B)** one equation **(C)** two equations
(D) three equations **(E)** all four equations

---

28. How many of the following equations represent parabolas?

$x^2 - y = 9$ $5x + y^2 = 4$ $x^2 + 1 = -y^2$ $x^2 - y = 0$

(A) none **(B)** one equation **(C)** two equations
(D) three equations **(E)** all four equations

---

29. The area of a triangle with base of length 3 and height (or altitude) of 10 is

(A) 13 **(B)** $\sqrt{13}$ **(C)** 15 **(D)** $\sqrt{30}$ **(E)** none of these

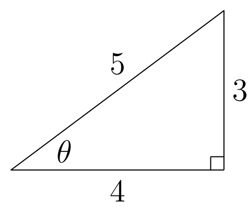
---

30. Which of the following curves passes through the points $(1,2)$ and $(2,-1)$?

(A) $x^2 - y^2 = 5$ **(B)** $x = y - 3$ **(C)** $y = 5 - 3x$ **(D)** $x^2 + y^2 = 3$ **(E)** none of these

---

31. In the following diagram $\tan\theta =$



(A) $\frac{5}{4}$ **(B)** $\frac{4}{5}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{3}{5}$ **(E)** none of these

---

32. Which of these is $\frac{\pi}{2}$ radians?

(A) 57.3° **(B)** $\frac{22^\circ}{7}$ **(C)** 90° **(D)** 180° **(E)** none of these

---

33. $\sin(60^\circ)$ is

(A) $\frac{\sqrt{3}}{2}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{1}{\sqrt{2}}$ **(D)** 1 **(E)** none of these

---

34. $\tan(-\pi)$ is

(A) -1 **(B)** 0 **(C)** 1 **(D)** undefined **(E)** none of these

---

35. If $\sin\theta = \frac{2}{5}$ and θ is in the first quadrant, then $\cos\theta =$

(A) $\frac{3}{5}$ **(B)** $\frac{\sqrt{21}}{5}$ **(C)** $\frac{\pi}{7}$ **(D)** $\frac{5}{3}$ **(E)** none of these

---

36. $\sin(2x) =$

(A) $2\sin(x)$ **(B)** $2\cos(x)\sin(x)$ **(C)** $\cos(x)\sin(x)$ **(D)** $\cos^2(x) - \sin^2(x)$ **(E)** none of these

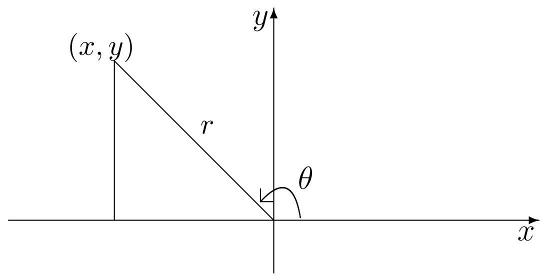
---

37. $\sin^2(\frac{\pi}{4}) - \cos^2(\frac{\pi}{4}) =$

(A) 1 **(B)** 0 **(C)** $\frac{2}{\sqrt{2}}$ **(D)** $\frac{\sqrt{2}}{2}$ **(E)** none of these

---

38. In the following diagram $\cot\theta =$



(A) $\frac{x}{r}$ **(B)** $\frac{x}{y}$ **(C)** $\frac{y}{r}$ **(D)** $\frac{y}{x}$ **(E)** none of these

---

39. For the equation $\cos^2(x) - 3\cos(x) + 2 = 0$ in the interval $[-\pi, \pi]$:

(A) there are no solutions
(B) there is exactly one solution
(C) there are exactly two solutions
(D) there are exactly three solutions
(E) none of the above is true

---

40. The equation $2^{2\sin x} + 2^{\sin x} - 6 = 0$ has

(A) only the solution $x = \frac{\pi}{2}$

(B) the solutions $x = \frac{\pi}{2} + k\pi,\quad k$ any integer

(C) the solutions $x = \frac{\pi}{2} + 2k\pi,\quad k$ any integer

(D) only the solution $x = \pi$

(E) the solutions $x = k\pi,\quad \pi$ any integer

Answer Key

#	Answer
1.	B
2.	E
3.	D
4.	B
5.	C
6.	C
7.	B
8.	D
9.	A
10.	D
11.	B
12.	D
13.	E
14.	C
15.	D
16.	E
17.	B
18.	A
19.	E
20.	E

#	Answer
21.	C
22.	A
23.	B
24.	E
25.	C
26.	D
27.	B
28.	D
29.	C
30.	C
31.	C
32.	C
33.	A
34.	B
35.	B
36.	B
37.	B
38.	B
39.	B
40.	C