SNAP Centre Workshop

Absolute Value

Introduction

The absolute value of some real number x (denoted by placing x within **absolute value bars**) is the non-negative value of x, regardless of x's sign.

We can define this mathematically using the equations:

$$|\boldsymbol{x}| = \sqrt{\boldsymbol{x^2}}$$

or

$$|\boldsymbol{x}| = \begin{cases} x \ if \ x \geq 0 \\ -x \ if \ x < 0 \end{cases}$$

So, the absolute value of a real number is the **non-negative root of that real number squared**.

Evaluating Absolute Values

Example 1 Find the absolute value of 9.

Using the absolute value equation, we can substitute x with 9 to find its absolute value.

$$|9| = \sqrt{9^2} = \sqrt{81} = \mathbf{9}$$

9 *and* $|9|$ *are equal, which makes sense, since* 9 *is already a non-negative value.*

Example 2 Find the absolute value of -9

Again, use the absolute value equation.

$$|-9| = \sqrt{(-9)^2} = \sqrt{81} = \mathbf{9}$$

As may have been expected, $|-9|$ *is equal to* 9*, which is the non-negative value of* -9*.*

Notice how important the correct order of operations is here. It didn't make a difference when finding the absolute value of 9, however, if we were to take the square root of -9 first, our final absolute value would have been an imaginary number squared, which would ultimately result in a negative number.

Example 3 Find the absolute value of 0.

$$|0| = \sqrt{0^2} = \sqrt{0} = \mathbf{0}$$

Although it may seem like a pointless exercise, noting that the absolute value of 0 is equal to 0 accounts for why we use the term "non-negative" in our absolute value definition, rather than "positive", as 0 is neither positive nor negative.

Example 4 Evaluate $-|-5|$.

Remembering the standard order of operations, we know to treat the absolute value bars as we would parentheses, meaning we evaluate the absolute value of -5 *first.*

$$-|-5|$$
$$= -(5)$$
$$= \mathbf{-5}$$

There is no need to continually substitute simple values into the absolute value equation in order to find their absolute value. By observation, we know that the non-negative value of -5 *is* 5.

Our final answer is -5 *since the negative sign* ***outside*** *the absolute value bars was applied after the absolute value of* -5 *is evaluated.*

If we ignored the order of operations and applied the negative sign outside the absolute value bars to the -5 inside the absolute value bars first, our answer would be incorrect.

$$-|-5|$$
$$= |-(-5)|$$
$$= |5|$$
$$= \mathbf{5}$$

Since the negative sign was applied to the term inside the absolute value bars prior to evaluating the absolute value, our final answer here is (incorrectly) 5.

This property can be generalized:

$$|-\boldsymbol{a}| \neq -|\boldsymbol{a}|$$

Example 5 $|7-10|$ $\quad Evaluate.$

Again, treat the absolute value bars as parentheses and perform the operation inside them first.

$$|7-10|$$
$$= |-3|$$
$$= \mathbf{3}$$

The operation within the absolute value bars resulted in a negative number. Taking the absolute value of this negative number resulted in a positive final answer.

Ignoring the correct order of operations in **Example 5** by taking the absolute value of each term first will result in an incorrect answer.

$$|7-10|$$
$$= |7+(-10)|$$

$= |7| + |-10|$

$= 7 + 10$

$= \mathbf{17}$

This property can be generalized:

$$|\boldsymbol{a} + \boldsymbol{b}| \neq |\boldsymbol{a}| + |\boldsymbol{b}|$$

Solving Absolute Value Equations

Solving for a given variable when absolute values are involved can be a little bit more complicated than evaluating an absolute value composed of constants. These complications are due to the need to account for both non-negative and negative values that have the same absolute value.

Example 6 $\quad |t| = 6 \quad$ $Solve\ for\ t.$

We can solve for t by going back to our original absolute value equation.

$$|t| = \sqrt{t^2} = 6$$

Our goal is to isolate t. To do this, we begin by squaring our terms on both sides of the equation.

$$(\sqrt{t^2})^2 = 6^2$$

$$t^2 = 36$$

Left with t^2, we know t needs to be equal to the positive and negative roots of 36.

$$t = \pm\sqrt{36} = \pm\mathbf{6}$$

Our solution indicates two possible values for t; 6 and -6. We can check whether or not both values are part of our solution by substituting them back into the original equation.

For $t = 6$:

$$|t| = |6| = 6$$

For $t = -6$:

$$|t| = |-6| = 6$$

Using either value results in a true statement, verifying that our solution of $t = \pm 6$ is correct.

Generally, we can say:

$$\boldsymbol{If}\ |\boldsymbol{x}| = \boldsymbol{a}, \boldsymbol{then}\ \boldsymbol{x} = \pm\boldsymbol{a}$$

Note: This is only true if a is a non-negative value.

Example 6 $\quad Replace\ t\ in\ the\ last\ example\ with\ p - 3, then\ solve\ for\ p.$

We can start by replacing t with $p-3$ in our original equation, remembering to retain the absolute value bars.

$$|t| = |p-3| = 6$$

From here, we could go back to the absolute value equation to solve for p, however, we're able to save ourselves some time by remembering that $t = \pm 6$, allowing us to conclude that $p-3 = \pm 6$.

To find a solution, we need to solve for p in both cases.

For $p-3=6$:

$$p-3=6$$

$$\boldsymbol{p=9}$$

For $p-3=-6$:

$$p-3=-6$$

$$\boldsymbol{p=-3}$$

We find that p has two distinct values, 9 and -3.

It may be tempting to try to isolate p first then take the absolute value, however, this would result in an incorrect solution:

$$|p-3| = 6$$

Ignoring the correct order of operations, we will add 3 to both sides to isolate $|p|$.

$$|p| = 9$$

$$p = \pm 9$$

One of our two values is still correct, however, the -9 found using this method is not part of the correct solution, and the -3 that should be part of our solution was not found at all.

Example 7 $\frac{12+2|3x-4|}{2} = 18 - x$ *Solve for x.*

Normally, our first step when solving for an unknown variable would be isolating the variable. Instead, we will be isolating the absolute value expression first.

$$12 + 2|3x-4| = 36 - 2x$$

$$2|3x-4| = 24 - 2x$$

$$|3x-4| = 12 - x$$

*With the absolute value expression isolated, we continue the same way we did in **Example 6;** we need to solve for x in two different cases.*

$$3x - 4 = \pm(12 - x)$$

For $3x-4 = 12-x$:

$3x - 4 = 12 - x$

$3x + x = 12 + 4$

$4x = 16$

$\boldsymbol{x = 4}$

For $3x - 4 = -(12 - x)$:

$3x - 4 = -(12 - x)$

$3x - 4 = -12 + x$

$3x - x = -12 + 4$

$2x = -8$

$\boldsymbol{x = -4}$

We find that x has two values, 4 and -4. Plugging either of these values back into our original absolute value equation confirms our solution.

Example 8 $|w - 7| + 10 = 2$ $\quad Solve\ for\ w.$

We begin by isolating our absolute value.

$|w - 7| = -8$

*Normally we would examine the negative and non-negative possibilities and solve for w accordingly, however, by noting that our absolute value expression is equal to a negative value, we know that there can be **no solution**. No value can be substituted into our equation for w that will result in the value -8.*

Example 9 $|p - 2| = 2p$

Our absolute value expression is already isolated, so we begin by examining each of our two possibilities.

For $p - 2 = 2p$*:*

$p - 2 = 2p$

$p - 2p = 2$

$-p = 2$

$\boldsymbol{p = -2}$

For $p - 2 = -2p$*:*

$p - 2 = -2p$

$p + 2p = 2$

$3p = 2$

$\boldsymbol{p = \frac{2}{3}}$

Although it appears as though we have found two possible values for p, plugging them back into our original equation proves otherwise. $p = \frac{2}{3}$ *results in a true statement, making it a valid solution.*

$p = -2$*, however, results in the following:*

$$|(-2) - 2| = 2(-4)$$

$$|-4| = -4$$

$$4 = -4$$

Since $4 \neq -4$*, we do not include* $p = -2$ *as part of our solution. As a general rule, check your solution by plugging your values back into your original equation.*

Example $|9 - 2r| = |3r + 12|$ $\quad Solve\ for\ r.$

When presented with an absolute value expression on both sides of the equation, we need to consider the following possibilities:

$$9 - 2r = 3r + 12$$

$$-(9 - 2r) = 3r + 12$$

$$9 - 2r = -(3r + 12)$$

However, since the second and third equations listed are essentially the same, there are really only two possibilities we need to consider, making the process of solving this type of absolute value equation the same as any other.

$$9 - 2r = \pm(3r + 12)$$

For $9 - 2r = 3r + 12$:

$$9 - 2r = 3r + 12$$

$$9 - 12 = 3r + 2r$$

$$-3 = 5r$$

$$-\boldsymbol{\frac{3}{5}} = \boldsymbol{r}$$

For $9 - 2r = -(3r + 12)$*:*

$$9 - 2r = -(3r + 12)$$

$$9 - 2r = -3r - 12$$

$$9 + 12 = -3r + 2r$$

$$21 = -r$$

$$\boldsymbol{-21 = r}$$

Plugging our values into our original equation confirms that they form a solution.